Lecture 10 (1/26/22).

We are proving:

Arzela-Ascoli Thm. A family $\mathcal{F} \subseteq C(G,\Omega)$ is normal $\Leftrightarrow$ TFH:

(i) $\forall z \in G$, $\{f(z) : f \in \mathcal{F}\}$ compact.

(ii) $\mathcal{F}$ is equicont. at every $z \in G$.

Pf. We started "$\Rightarrow$" and proved $\mathcal{F}$ normal $\Rightarrow$ (i). To prove that also (ii) holds, we recall: $\mathcal{F}$ normal $\Leftrightarrow$ $\overline{\mathcal{F}}$ compact and (X,d) is compact $\Leftrightarrow$ (X,d) is complete and totally bounded, i.e. $\forall \varepsilon > 0$ $\exists x_1, \dots, x_N$ s.t. $X = \bigcup_{n=1}^{N} B(x_n, \varepsilon)$. We now show:

(ii). Fix $z_0 \in G$ and pick $\varepsilon > 0$. Since $\overline{\mathcal{F}}$ is compact $\Rightarrow$ $\overline{\mathcal{F}}$ is totally bdd $\Rightarrow$ (HW) $\mathcal{F}$ is totally bdd. Let $\alpha_0 > 0$ s.t. $K = \overline{B(z_0; \alpha_0)} \subset\subset G$. By Lemma 1, last lecture

$\exists\, \delta > 0$ s.t. $\rho(f,g) < \delta \Rightarrow \sup_K d(f,g) < \frac{\varepsilon}{3}$.

- $\exists\, f_1, \dots, f_N \in \mathcal{F}$ s.t. $\mathcal{F} \subseteq \bigcup_{n=1}^{N} B(f_n, \delta)$.

- $\exists\, 0 < \alpha < \alpha_0$ s.t. $|z - z_0| < \alpha \Rightarrow d(f_n(z), f_n(z_0)) < \frac{\varepsilon}{3}$ for $n = 1, \dots, N$. But then for any $f \in \mathcal{F}$, $f \in B(f_n, \delta)$ for some n $\Rightarrow$

$$d(f(z), f(z_0)) \le d(f(z), f_n(z)) + d(f_n(z), f_n(z_0))$$
$$+ d(f_n(z_0), f(z_0)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$\Rightarrow$ (ii) holds.

$\Leftarrow$ We shall show $\mathcal{F}$ normal by using Tychanoff. Thus:

let $\{z_n\}_{n=1}^{\infty}$ be a countable subset of G that is dense (e.g., points in G w/ rational real <u>and imaginary</u> parts), consider $X_n = \{f(z_n) : f \in \mathcal{F}\} \subseteq \mathbb{C}$, and $X = \prod_{n=1}^{\infty} X_n$. By (i), each X_n is compact, so by Tychanoff, X is compact.

We shall show that every seq. $\{f_k\}$ in $\mathcal{F}$ has a convergent subsequence $\{f_{k_j}\}$.

$\{f_k\}$ gives rise to a seq $\{x^k\}$ in X by $x^k_n = f_k(z_n)$. By compactness, $\exists$ convergent subseq. x^{k_j}. $\Rightarrow \exists\, x = \{\xi_n\}_n$ in $\mathfrak{X}$ s.t. $f_{k_j}(z_n) \xrightarrow[j\to\infty]{} \xi_n$. We shall show $\{f_{k_j}\}$ is Cauchy on compacts. Let $K \subset\subset G$, $\varepsilon > 0$. Need:

<u>Prop1.</u> If $\mathcal{F}$ is equicont. at every $z \in G$, then $\mathcal{F}$ is unif. equicontinuous on every compact $K \subset\subset G$ (i.e. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall z \in K$, $\forall f \in \mathcal{F}$, $d(f(z), f(z')) < \varepsilon$ if $|z - z'| < \delta$).

Pf. Same pf as showing that one cont. fcn on G is unif. cont. on compacts. Details are DIY. $\boxtimes$

Let $\delta > 0$ be the one given by Prop1 for K and $\varepsilon/3$ (uniform equicont. on K).

Clearly, $\{B(z_n, \delta)$ for $z_n \in K\}$ is an open cover of $K \Rightarrow$ (after renumbering the z_n, if necessary) $K \subseteq \bigcup_{n=1}^{N} B(z_n, \delta)$.

Choose $J \in \mathbb{N}$ so that $d(f_{k_j}(z_n), f_{k_\ell}(z_n)) < \varepsilon/3$ for $j, \ell \geq J$ and $n \in \{1, \dots, N\}$.

For any $z \in K$, $z \in B(z_n, \delta)$ for some $n \in \{1, \dots, N\}$, $\Rightarrow$ for $j, \ell \geq J$,

$$d(f_{k_j}(z), f_{k_\ell}(z)) \overset{\Delta\text{-ineq.}}{\leq} d(f_{k_j}(z), f_{k_j}(z_n)) + d(f_{k_j}(z_n), f_{k_\ell}(z_n)) + d(f_{k_\ell}(z_n), f_{k_\ell}(z))$$

$$\underset{\text{equicont. at } z_n \text{ + conv. at } z_n}{<} \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

Thus, we have shown that the subseq. $\{f_{k_j}\}_j$ is unif. Cauchy on every compact $K \subset\subset G \Rightarrow$ $\{f_{k_j}\}_j$ is Cauchy in $\mathcal{C}(G, \Omega) \Rightarrow$

convergent by completeness of $\mathcal{C}(G,\Omega)$.

$\boxtimes$

Return to complex analysis:

Space of analytic functions.

Given $G \subseteq \mathbb{C}$ open, $H(G)$ denotes the space of analytic (holomorphic) functions in G. Clearly, $H(G) \subseteq \mathcal{C}(G,\mathbb{C})$.
(Note: In literature, $\mathcal{O}(G)$ is often used.)

Thm 1. If $\{f_k\}_k$ is a seq. in $H(G)$ s.t. $f_k \to f$ in $\mathcal{C}(G,\mathbb{C})$, then $f \in H(G)$. Moreover, $f_k^{(n)} \to f^{(n)}$ for all n.

Pf. $f_k \to f$ in $\mathcal{C}(G,\mathbb{C}) \Leftrightarrow f_k \to f$ unif. on compacts $\Rightarrow f$ is analytic by Morera and Cauchy's Thm. DIY.
Similarly, $f_k^{(n)} \to f^{(n)}$ for all n by Cauchy's estimate (or formula) as follows:

Let $K \subset\subset G$, and $\delta = d(K, \mathbb{C} \setminus G) > 0$.
Let $U = \bigcup_{z \in K} B(z, \delta/2)$. Then $K \subset\subset U$,

$\overline{U} \subset\subset G$, $d(K, \mathbb{C} \setminus \overline{U}) = \delta' \geq \delta/2$

By Cauchy's Estimate in $\overline{U}$:

$$\sup_K |f^{(n)} - f_{k_j}^{(n)}| \leq \frac{n!}{(\delta')^n} \sup_{\overline{U}} |f - f_{k_j}|.$$

Since $f - f_{k_j} \to 0$ unif. on $\overline{U}$, $f^{(n)} - f_{k_j}^{(n)} \to 0$ unif. on K. $\Rightarrow f_{k_j}^{(n)} \to f^{(n)}$ in $C(G, \mathbb{C})$. $\boxtimes$

<u>Cor1.</u> $H(G)$ is closed in $C(G, \mathbb{C})$.